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Some Properties of Fixed Point for Contraction Mappings in Quasi ????b-metric Space

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Some Properties of Fixed Point for Contraction Mappings in Quasi αb-metric Space

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Abstract
We propose fixed point theorems for some contraction mappings in quasi αb-metric space. Especially, the sufficient conditions to obtain an existence and uniqueness of fixed point on certain contraction mappings. The quasi αb-metric space is extension of quasi b-metric space.

1. Introduction

We know that b-metric space was expressed by Bakhtin in 1989 [1], then Czewick in 1993 applied it for his study on contraction mappings in the b-metric space [2]. However, many authors used the b-metric space for existence and uniqueness of fixed point on several contraction mappings [3, 4, 5] and developed the b-metric space to become quasi b-metric space. Most of authors used the quasi b-metric space as a dislocated quasi b-metric space, and used it on fixed point theorems for several contraction mappings [6, 7, 8, 9]. While, quasi αb-metric space was developed from quasi b-metric space, with modifying triangular inequality condition of quasi b-metric. This work is motivated by the results on fixed point for contraction mappings in b-metric space, especially in creating of sufficient condition for existence and uniqueness of fixed point in the quasi αb-metric space [10].

For showing some properties of quasi αb-metric space for fixed point on contraction mapping, it is needed some notions as follows.

2. Preliminaries

Definition 2.1. [1, 2] *Let X be a nonempty set and b ≥1.*

Let d : X × X → [0, ∞) be and for all $x, y, z \in X$ *the following conditions are satisfied:*

(1) $d(x,y) = d(y,x) = 0$ *if and only if* $x = y$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,y) \le b\big(d(x,z) + d(z,y)\big)$.

Then d is called as b-metric on X and (X,d) is called as b-metric space. And (X,d) is called as quasi b-metric space if only (1), (3) hold.

Definition 2.2.[10] *Let X be a nonempty set,* $0 \le \alpha < 1$ *and* $b \ge 1$.

Let d : X × X → [0, ∞) be a mapping which satisfying the conditions:

(1) $d(x,y) = d(y,x) = 0$ *if and only if* $x = y$;
(2) $d(x,y) \le \alpha d(y,x) + \frac{1}{2} b\big(d(x,z) + d(z,y)\big)$ *for all* $x, y, z \in X$ (1)

Then d is called as a *quasi αb-metric* on X and (X,d) is called as a *quasi αb-metric space*.

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Example 2.4.[10] *Let* $X = \{0,1,2\}$. *Defined* $d: X \times X \to R^+$ *as follows* $d(0,0) = d(1,1) = d(2,2) = d(0,2) = d(2,1) = 0$, $d(1,0) = 4$, $d(2,0) = 1$, $d(0,1) = 2$, *and* $d(1,2) = 3$. *It shows that d is quasi* αb*-metric with* $\alpha = \frac{1}{2}$ *and* $b = 4$, *because* $2 = d(0,1) \le \frac{1}{2} d(1,0) + 2(d(0,2) + d(2,1))$, *but* $2 = d(0,1) > c(d(0,2) + d(2,1))$ *for every* $c \ge 1$, *so d is not quasi b-metric.*

It shows that quasi αb-metric space is not necessarily a quasi b-metric.

Example 2.5. [10] *Let* $X = R$ *and defined* $d: X \times X \to R^+$ *as* $d(x,y) = \begin{cases} 2x^2 + y^2, & x \ne y \\ 0, & x = y \end{cases}$

For the first condition of quasi αb*-metric is obvious from above definition of function d(x,y), and for the second condition, we can show that*
For $x \ne y$, *and every* $z \in X$, *we have*

$$d(x,y) = 2x^2 + y^2 \le \frac{5}{2}x^2 + 2y^2 + 3z^2 = \frac{1}{2}(2y^2 + x^2) + ((2x^2 + z^2) + (2z^2 + y^2))$$

$$= \frac{1}{2} d(y,x) + \frac{2}{2}(d(x,z) + d(z,y))$$

So we obtain $d(x,y) = \frac{1}{2} d(y,x) + \frac{2}{2}(d(x,z) + d(z,y))$
Hence d is a quasi αb*-metric with* $\alpha = \frac{1}{2}$ *and* $b = 2$.

Definition 2.6.[10] *Let* (X,d) *be a quasi* αb*-metric space, a sequence* $\{x_n\}$ *in* (X,d) *is called convergent to* $x \in X$ *if* $\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,x_n) = 0$, *we write* $\lim_{n\to\infty} x_n = x$.

Definition 2.7.[10] *Let* $\{x_n\}$ *be a sequence in quasi* αb*-metric space* (X,d), $\{x_n\}$ *is called Cauchy sequence if* $\lim_{n,m\to\infty} d(x_n,x_m) = \lim_{n\to\infty} d(x_m,x_n) = 0$

Definition 2.8. Let (X,d) *be a quasi* αb*-metric space,* (X,d) *is called complete if every Cauchy sequence in X is convergent in X .*

Definition 2.9. Let X be a nonempty set and let T be a self mapping on X, $x \in X$ *is called a fixed point of T, if* $Tx = x$. *And for every* $x \in X$ *be defined* $TT^{n-1}x = T^n x$ *with* $T^0 x = x$

For proving next theorems in main results, we need sufficient conditions that a sequence to be a Cauchy sequence in quasi αb-metric space as follows

Theorem 2.10. [10] *Let (X,d) be a quasi* αb*-metric space with* $0 \le \alpha < 1$ *and* $b \ge 1$, *and* $\{x_n\}$ *be a sequence in X satisfying the conditions as follows:*

1. $d(x_n, x_{n+1}) \le a d(x_{n-1}, x_n)$, *where* $0 < a < 1$;
2. $d(x_{n+1}, x_n) \le c d(x_n, x_{n-1})$, *where* $0 < c < 1$;
3. $ba + \alpha^2 < 1$ *and* $bc + \alpha^2 < 1$.

Then $\{x_n\}$ *is a Cauchy sequence in X*

3. Main Results

Some properties of existence and uniqueness of fixed point in quasi αb-metric space are shown on some theorems for certain contraction mappings as follows.

Theorem 3.1. Let (X, d) be a quasi αb-metric space with $0 \le \alpha < 1$ and $b \ge 1$. Then for every $x, y, z \in X$ holds

1) $d(x,y) \le \frac{b}{2(1-\alpha^2)}[(d(x,z) + d(z,y)) + \alpha(d(y,z) + d(z,x))]$
2) $d(x,y) + d(y,x) \le \frac{b}{2(1-\alpha)}(d(x,z) + d(z,y) + d(y,z) + d(z,x))$

Proof. For every $x, y, z \in X$, and from (1) we have

$$d(x,y) \le \alpha\, d(y,x) + \frac{1}{2}b\,(d(x,z) + d(z,y))$$
$$\le \alpha\left(\alpha\, d(x,y) + \frac{1}{2}b\,(d(y,z) + d(z,x))\right) + \frac{1}{2}b\,(d(x,z) + d(z,y))$$
$$= \alpha^2\, d(x,y) + \frac{1}{2}\alpha b\,(d(y,z) + d(z,x)) + \frac{1}{2}b\,(d(x,z) + d(z,y))$$

Thus,

$$d(x,y) \le \frac{b}{2(1-\alpha^2)}[(d(x,z) + d(z,y)) + \alpha(d(y,z) + d(z,x))] \quad (2)$$

Furthermore, from (1) we have

$$d(y,x) \le \frac{b}{2(1-\alpha^2)}[(d(y,z) + d(x,y)) + \alpha(d(x,z) + d(y,x))] \quad (3)$$

Therefore from (2) and (3) we obtain

$$d(x,y) + d(y,x) \le \frac{b}{2(1-\alpha^2)}[(1+\alpha)(d(x,z) + d(z,y)) + (1+\alpha)(d(y,z) + d(z,x))]$$
$$= \frac{(1+\alpha)b}{2(1-\alpha^2)}[(d(x,z) + d(z,y)) + (d(y,z) + d(z,x))]$$
$$= \frac{b}{2(1-\alpha)}[(d(x,z) + d(z,y)) + (d(y,z) + d(z,x))]$$

It implies that part 2) is true □

Theorem 3.2. Let (X, d) be a complete quasi αb-metric space with $0 \le \alpha < 1$ and $b \ge 1$. $T: X \to X$ be a continuous mapping which satisfying the conditions

$$d(Tx, Ty) \le \frac{pq\, d(x,Tx)\, d(Ty,y)}{p\, d(x,Tx) + q\, d(Ty,y) + r} \quad (4)$$

For every $x, y \in X$, $0 < p < 1$, $0 < q < 1$, $r > 0$, $bp + \alpha^2 < 1$ and $bq + \alpha^2 < 1$. Then there exists a unique fixed point of T in X.

Proof. First, we will show that T has a fixed point in X. Taken $x_0 \in X$ and made a sequence $\{x_n\}$ where $x_1 = Tx_0$ and $x_{n+1} = Tx_n$ for $n = 0,1,2,\ldots$. Then using (4) we get

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \le \frac{pq\, d(x_{n-1},x_n)\, d(x_{n+1},x_n)}{p\, d(x_{n-1},x_n) + q\, d(x_{n+1},x_n) + r}$$
$$< \frac{pq\, d(x_{n-1},x_n)\, d(x_{n+1},x_n)}{q\, d(x_{n+1},x_n)} = pd(x_{n-1},x_n)$$

So we have

$$d(x_n, x_{n+1}) < p\, d(x_{n-1}, x_n) \quad (5)$$

With similarly way and using (5) we get

$$d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1}) \leq \frac{pq\, d(x_n,x_{n+1})\, d(x_n,x_{n-1})}{p\, d(x_n,x_{n+1}) + q\, d(x_n,x_{n-1}) + r} < \frac{pq\, d(x_n,x_{n+1})\, d(x_n,x_{n-1})}{p\, d(x_n,x_{n+1})}$$

So we have

$$d(x_n,x_{n+1}) < q\, d(x_n,x_{n-1}) \quad (6)$$

So from (5) and (6) we obtain that

$d(x_{n+1},x_n) < p\, d(x_{n-1},x_n)$,

$d(x_n,x_{n+1}) < q\, d(x_n,x_{n-1})$

Since $bp + \alpha^2 < 1$, $bq + \alpha^2 < 1$, and according to *Theorem 2.10*, then we conclude that $\{x_n\}$ is a Cauchy sequence in X. Since X complete then there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

Since T continuous on X, then

$Tx^* = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*$.

Hence, x^* is the fixed point of T.

To show the uniqueness of fixed point of T, we suppose that there exists $y^* \in X$ such that $Ty^* = y^*$. Then from (4), we have

$$d(x^*,y^*) = d(Tx^*,Ty^*) \leq \frac{pq\, d(x^*,Tx^*)\, d(Ty^*,y^*)}{p\, d(x^*,Tx^*) + q\, d(Ty^*,y^*) + r} = \frac{pq\, d(x^*,x^*)\, d(y^*,y^*)}{p\, d(x^*,x^*) + q\, d(y^*,y^*) + r} = 0$$

This implies $d(x^*,y^*) = 0$. And using similarly way we have $d(y^*,x^*) = 0$. So we obtain

$x^* = y^*$

Thus T has a unique fixed point in X □

Theorem 3.3. *Let* (X,d) *be a complete quasi* $\alpha b-$*metric space with* $0 \leq \alpha < 1$ *dan* $b \geq 1$. *Let* $T: X \to X$ *be a continuous function satisfies the conditions*

$$d(Tx,Ty) \leq \min\{pd(x,Tx), q\, d(Ty,y)\} \quad (7)$$

for every $x,y \in X$, $0 < p < 1$, $0 < q < 1$, $bp + \alpha^2 < 1$, $bq + \alpha^2 < 1$.
Then T *has a unique fixed point in* X.

Proof. Taken $x_0 \in X$, $x_1 = Tx_0$ and $x_{n+1} = Tx_n$ for $n = 0,1,2,\ldots$.
So $\{x_n\}$ is a sequence in X, then using (7) we obtain

$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \leq \min\{p\, d(x_{n-1},x_n), q\, d(x_{n+1},x_n)\} \leq p\, d(x_{n-1},x_n)$

So we have

$$d(x_n,x_{n+1}) \leq p\, d(x_{n-1},x_n) \quad (8)$$

Once again, we use condition (7) then we obtain

$d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1}) \leq \min\{p\, d(x_n,x_{n+1}), q\, d(x_n,x_{n-1})\} \leq q\, d(x_n,x_{n-1})$

So we have

$$d(x_{n+1},x_n) \leq q\, d(x_n,x_{n-1}) \quad (9)$$

From (8) and (9) , since $p + \alpha^2 < 1$, $bq + \alpha^2 < 1$ and using *Theorem 2.10*, thus we have $\{x_n\}$ is a Cauchy sequence in X. Since X complete, thus we have $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

And since T continuous in X, then we obtain

$Tx^* = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*$.

Hence x^* is the fixed point of T.
To show x^* as an unique fixed point of T, we suppose there exists $y^* \in X$ such that $Ty^* = y^*$.
From condition (7) we can obtain

$$\begin{aligned} d(x^*,y^*) = d(Tx^*,Ty^*) &\leq \min\{p\, d(x^*,Tx^*), q\, d(Ty^*,y^*)\} \\ &= \min\{p\, d(x^*,x^*), q\, d(y^*,y^*)\} \\ &= 0 \end{aligned}$$

This implies $d(x^*,y^*) = 0$. Similarly we have also $d(y^*,x^*) = 0$. This implies that $x^* = y^*$.
Thus T has a unique fixed point in X. □

Theorem 3.4. *Let* (X,d) *be a complete quasi* αb*-metric space with* $0 \leq \alpha < 1$ *and* $b \geq 1$. *Let* $T: X \to X$ *be a continuous function satisfies conditions as follows*

$$d(Tx,Ty) \leq k\, d(x,y) + d(Tx,y) d(x,Ty) d(x,Tx) \qquad (10)$$

for every $x,y \in X$, *where* $0 < k < 1$ *and* $bk + \alpha^2 < 1$.
Then there exists an element in X *as a unique fixed point of* T.

Proof. Let $x_0 \in X$ and $x_1 = Tx_0$ and $x_{n+1} = Tx_n$ for $n = 0,1,2,\ldots$, then $\{x_n\}$ is a sequence in X.
From condition (10) we obtain

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\ &\leq k\, d(x_{n-1},x_n) + d(Tx_{n-1},x_n)\, d(x_{n-1},Tx_n)\, d(x_{n-1},Tx_{n-1}) \\ &= k\, d(x_{n-1},x_n) + d(x_n,x_n)\, d(x_{n-1},x_{n+1})\, d(x_{n-1},x_n) \\ &= k\, d(x_{n-1},x_n) \end{aligned}$$

So, we have

$$d(x_n,x_{n+1}) \leq k\, d(x_{n-1},x_n) \qquad (11)$$

And then once again we use condition (10), so we get

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1}) \\ &\leq k\, d(x_n,x_{n-1}) + d(Tx_n,x_{n-1})\, d(x_n,Tx_{n-1})\, d(x_n,Tx_n) \\ &= k\, d(x_n,x_{n-1}) + d(x_{n+1},x_{n-1})\, d(x_n,x_n)\, d(x_n,Tx_{n+1}) \\ &= k\, d(x_n,x_{n-1}) \end{aligned}$$

So, we have

$$d(x_{n+1},x_n) \leq k\, d(x_n,x_{n-1}) \qquad (12)$$

And from using (11) and (12) we obtain
$d(x_n,x_{n+1}) \leq k\, d(x_{n-1},x_n)$ and $d(x_{n+1},x_n) \leq k\, d(x_n,x_{n-1})$
Since $0 < k < 1$, $bk + \alpha^2 < 1$ and using *Theorem 2.10*, this implies that $\{x_n\}$ is a Cauchy sequence in X.
Since X is a complete quasi αb-metric space, then there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.
Since T continuous on X then
$Tx^* = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*$.
To show the uniqueness of fixed point of T, it can be shown as follows
We Suppose that there exists $y^* \in X$ such that $Ty^* = y^*$. From condition (10) we obtain

$$\begin{aligned} d(x^*,y^*) = d(Tx^*,Ty^*) &\leq k\, d(x^*,y^*) + d(Tx^*,y^*)\, d(x^*,Ty^*)\, d(x^*,Tx^*) \\ &= k\, d(x^*,y^*) + d(x^*,y^*)\, d(x^*,y^*)\, d(x^*,x^*) \end{aligned}$$

Then we have

$$d(x^*,y^*) \leq k\, d(x^*,y^*) \qquad (13)$$

Since $0 < k < 1$ and from (13) then it is possible if $d(x^*,y^*) = 0$.
Similarly, using (10) we can have

$$d(y^*,x^*) \leq k\, d(y^*,x^*) \qquad (14)$$

And since $0 < k < 1$ and from (14) then is possible if $d(y^*,x^*) = 0$.

So we have $(x^*, y^*) = d(y^*, x^*) = 0$, it concludes that $x^* = y^*$. Thus T has a unique fixed point in X□.

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